$\boxed{1}$ Verify Stokes' thm for $F = \langle y, x, x^2+y^2 \rangle$ where S is the upper hemisphere $x^2+y^2+z^2=1$, $z \geq 0$ with an upward-pointing normal vector.

We want to show: $\underbrace{\iint_S \text{curl}(F)\cdot dS}_{(1)} = \underbrace{\oint_{\partial S} F \cdot dr}_{(2)}$

computing (1): $\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x & x^2+y^2 \end{vmatrix} = (\ 2y\ )i - (\ 2x\ )j + (\ 0\ )k$

$= \langle 2y, -2x, 0 \rangle.$

A parametrization for S is:

$G(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$ $\quad 0 \leq \theta \leq 2\pi$, $\quad 0 \leq \phi \leq \pi/2$

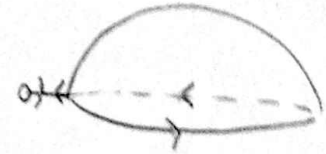
$N(\theta,\phi) = \sin\phi\,(\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$

So,

$$\iint_S \text{curl}(F)\cdot dS = \int_0^{\pi/2}\int_0^{2\pi} \langle 2\sin\theta\sin\phi, -2\cos\theta\sin\phi, 0\rangle \cdot \langle \cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi \rangle \sin\phi \, d\theta\, d\phi$$

$$= \int_0^{\pi/2}\int_0^{2\pi} \sin\phi\,(2\sin^2\phi\cos\theta\sin\theta) - \sin\phi\, 2\cos\theta\sin\theta\sin^2\phi \, d\theta\, d\phi = 0.$$

Computing (2): The boundary, ∂S is a unit circle oriented counterclockwise.



So, a parametrization for ∂S is:

$c(t) = (\cos t, \sin t, 0) \qquad 0 \le t \le 2\pi$

$c'(t) = (-\sin t, \cos t, 0)$

And $\quad F(c(t)) = \langle \sin t, \cos t, 1 \rangle$.

So, $$\oint_{\partial S} F \cdot dr = \int_0^{2\pi} F(c(t)) \cdot c'(t)\, dt = \int_0^{2\pi} \langle \sin t, \cos t, 1\rangle \cdot \langle -\sin t, \cos t, 0\rangle\, dt$$

$$= \int_0^{2\pi} -\sin^2 t + \cos^2 t\, dt = \int_0^{2\pi} \cos 2t\, dt = \frac{\sin 2t}{2}\Big|_0^{2\pi} = 0.$$

So, $(1) = (2) = 0 \Rightarrow$ Stokes' thm is verified for this example.

$\boxed{2}$ Calculate curl(F) and then apply Stokes' thm to compute the flux of curl(F) through the surface given. $F = \langle yz, -xz, z^3 \rangle$, the part of the cone $z = \sqrt{x^2+y^2}$ that lies between the two planes $z=1$ and $z=3$ with normal vector pointing outside the cone.

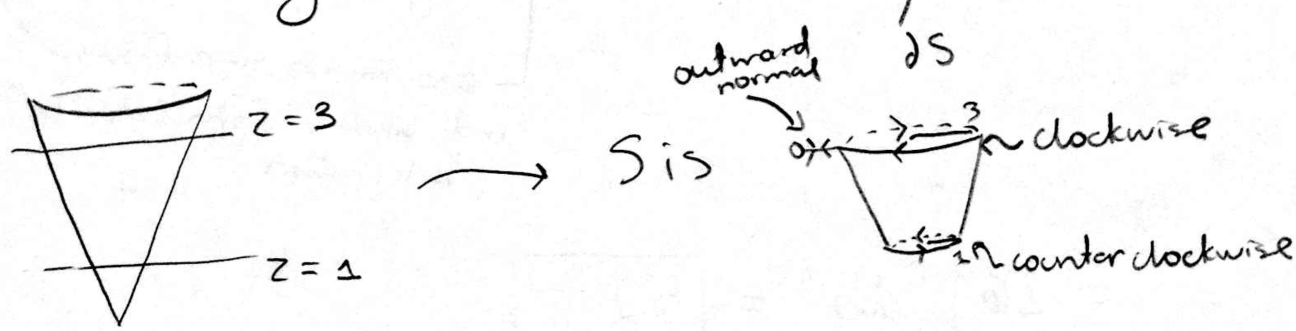
<u>Finding curl(F)</u>: $\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ yz & -xz & z^3 \end{vmatrix} = x\,i - y\,j - z\,k$

$= \langle x, -y, -2z \rangle$.

Stokes' thm say:

$$\iint_S \text{curl}(F) \cdot dS = \oint_{\partial S} F \cdot dr.$$

So, we identify ∂S and compute $\oint_{\partial S} F \cdot dr$.



So, $\partial S = C_3 + C_1$ where C_1 and C_3 are as in the sketch above.

C_1 is parametrized:

$r_1(\theta) = (\cos\theta, \sin\theta, 1)$ $\qquad 0 \le \theta \le 2\pi$

C_2 is parametrized:

$r_2(\theta) = (3\cos\theta, 3\sin\theta, 3)$ $\qquad 2\pi \le \theta \le 0$

switch order for clockwise

So, $\displaystyle\oint_{\partial S} F \cdot dr = \oint_{C_1} F \cdot dr + \oint_{C_3} F \cdot dr$

$$= \int_0^{2\pi} \langle \sin\theta, -\cos\theta, 1\rangle \cdot \langle -\sin\theta, \cos\theta, 0\rangle \, d\theta \; +$$

$$\int_{2\pi}^{0} \langle 9\sin\theta, -9\cos\theta, 9\rangle \cdot \langle -3\sin\theta, 3\cos\theta, 0\rangle \, d\theta$$

$$= -\int_0^{2\pi} \sin^2\theta + \cos^2\theta \, d\theta + 27\int_0^{2\pi} \sin^2\theta + \cos^2\theta \, d\theta$$

one minus sign from and another from switch order of integration

$$= 26\int_0^{2\pi} d\theta = \boxed{52\pi}$$

[3] Apply Stokes' thm to evaluate

$\oint_C F \cdot dr$ by finding the flux of curl(F)

across an appropriate surface for
$F = \langle y, -2z, 4x \rangle$, C is the boundary of the
plane $x + 2y + 3z = 1$ that is in the first
octant of space, oriented counterclockwise as
viewed from above.

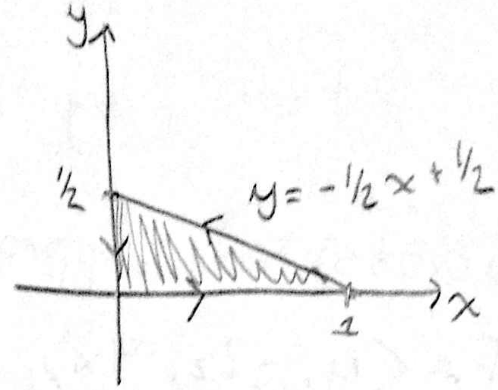
Finding curl(F): $\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -2z & 4x \end{vmatrix} = 2i - 4j - 1k$
$= \langle 2, -4, -1 \rangle$.

Stokes thm tells us that

$$\iint_S \text{curl}(F) = \oint_{\partial S} F \cdot dr.$$

So, we need to find a surface with boundary C.
We can take S to be the plane $x + 2y + 3z = 1$
with $x, y, z \geq 0$ then, a parametrization is:
$G(x,y) = \left(x, y, \frac{1-x-2y}{3}\right)$ with $x \geq 0$, $y \geq 0$
and $\frac{1-x-2y}{3} \geq 0$ ↳

The domain of the parametrization simplifies to:



So, $\iint_S curl(F) \cdot dS =$

$$\int_0^1 \int_0^{1/2(1-x)} \langle 2, -4, -1 \rangle \cdot N(x,y)\, dy\, dx.$$ But,

$$G_x \times G_y = \langle 1, 0, -1/3 \rangle \times \langle 0, 1, -2/3 \rangle$$
$$= \langle 1/3, 2/3, 1 \rangle$$

Since the boundary orientation is counterclockwise we want an upward normal.

So, $\displaystyle\int_0^1 \int_0^{1/2(1-x)} \langle 2,-4,-1 \rangle \langle 1/3, 2/3, 1 \rangle\, dy\, dx =$

$$\int_0^1 \int_0^{1/2(1-x)} 2/3 - 8/3 - 1\, dy\, dx = -3 \int_0^1 \int_0^{1/2(1-x)} dy\, dx$$

$$= \frac{-3}{2} \int_0^1 1 - x\, dx = \frac{-3}{2}\left[x - \frac{x^2}{2} \Big|_0^1 \right]$$

$$= \frac{-3}{2}\left[1 - \frac{1}{2}\right] = \boxed{-3/4}$$

$\boxed{4}$ Let S be the portion of the plane $z=x$ contained in the half cylinder of radius R depicted in the Figure. Use Stokes' thm to calculate the circulation of $F=\langle z, x, y+2z\rangle$ around the boundary of S (a half ellipse) in the counter clockwise direction when viewed from above.

Stokes' thm tells us that

$$\oint_{\partial S} F \cdot dr = \iint_S \text{curl}(F) \cdot dS.$$

↑we want this so, we find $\iint_S \text{curl}(F) \cdot dS$.

<u>Finding curl(F)</u>: $\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ z & x & y+2z \end{vmatrix} = 1i + 1j + 1k = \langle 1, 1, 1\rangle.$

A normal vector of the plane is:

$z=x \Rightarrow -x+z=0 \Rightarrow N=\langle -1, 0, 1\rangle.$

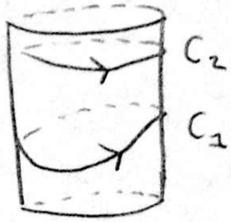
So, $\iint_S \text{curl}(F) \cdot dS = \frac{1}{\sqrt{2}} \iint_S \langle 1,1,1\rangle \cdot \langle -1,0,1\rangle \, dS = \underline{0}.$

So, we got our answer w/o parametrizing!

$\boxed{5}$ Let $F = \langle y^2, x^2, z^2 \rangle$. Show that

$$\int_{C_1} F \cdot dr = \int_{C_2} F \cdot dr$$ for any two closed curves

lying on the cylinder whose central axis is the z-axis.



Denote by S the part of the cylinder for which C_1 and C_2 are boundary curves. Lets Apply Stokes' thm to S w/ outward pointing normal. Then,

$$\oint_{\partial S} F \cdot dr = \oint_{C_1} F \cdot dr - \oint_{C_2} F \cdot dr = \iint_S \text{curl}(F) \cdot dS.$$

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & x^2 & z^2 \end{vmatrix} = 0i - 0j + (2x - 2y)k$$
$$= \langle 0, 0, 2x - 2y \rangle.$$

We parametrice S:

$G(\theta, z) = \langle R\cos\theta, R\sin\theta, z \rangle$ where (θ, z) are in an appropriate domain D [note we will not need to find D].

Then, $n(\theta, z) = \langle R\cos\theta, R\sin\theta, 0 \rangle$ So,

$$\text{curl}(F)(G(\theta, z)) \cdot n(\theta, z) = \langle 0, 0, 2R\cos\theta - 2R\sin\theta \rangle \cdot \langle R\cos\theta, R\sin\theta, 0 \rangle$$
$$= 0. \hookrightarrow$$

So, $\displaystyle\oint_{C_1} F\cdot dr - \oint_{C_2} F\cdot dr = 0 \Rightarrow$

$$\oint_{C_1} F\cdot dr = \oint_{C_2} F\cdot dr \quad \text{as needed.}$$

$\boxed{6}$ You know two things about a vector field F: (i) F has a vector potential A (A not known)

(ii) The circulation of A around the unit circle (oriented counter-clockwise) is 25.

Determine the Flux of F through S in Fig. 22 w/ upward pointing normal.

Stokes' thm tells us that:

$\displaystyle\iint_S \operatorname{curl}(A)\cdot dS = \oint_{\partial S} A\cdot dr = 25$. Since $F = \operatorname{curl}(A)$, we have

$$\text{Flux of } F \text{ through } S = \iint_S F\cdot dS = \iint_S \operatorname{curl}(A) = \boxed{25}.$$

$\boxed{7}$ Assume that f and g have continuous partial derivatives of order 2. Prove that

$$\oint_{\partial S} f\nabla(g)\cdot dr = \iint_S \nabla(f)\times\nabla(g)\cdot dS$$

$\hookrightarrow$

$\boxed{8}$ Explain carefully why Greens thm is a special case of stokes thm.

Let D be a region in the plane xy. Then we can view D as a surface S in $\mathbb{R}^3$ with normal vector $n = (0,0,1)$.

So, Stokes' thm tells us:

$$\int_{\partial D} F \cdot dr = \oint_{\partial S} F \cdot dr = \iint_S \mathrm{curl}(F) \cdot dS = \iint_S \left\langle \frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}, \frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}, \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right\rangle \cdot \langle 0,0,1 \rangle \, dS$$

$$= \iint_S \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \right) dS = \iint_D \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial x} \right) dA$$

$$= \iint_D \mathrm{curl}_z (F) \, dD \implies$$

$\int_{\partial D} F \cdot dr = \iint_D \mathrm{curl}_z (F) \, dD$. This is exactly the statement of Green's thm. So, Greens theorem is a special case of the more general Stokes thm.